Ginzburg-Landau
Handling the non-normality

Forced nonparallel, nonlinear Ginzburg-Landau equation:

$$\partial_t w + U\partial_x w + w|w|^2 = \mu(x)w + \gamma\partial_{xx}w + f(x,t)$$
$$\mu(x) := i\omega_0 + \mu_0 - \gamma\chi^4 x^2$$
$$|w| \to 0 \text{ as } \mathrm{x} \to \pm\infty$$

and $\mathrm{U}, \gamma,\ \omega_0, \mu_0, \chi$ are positive real constant, $f(x,t)$ a “weak” forcing

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Ginzburg-Landau
Change of variables

We consider the following change of varibles:

$$u = \gamma\chi^2 t$$
$$y = \chi x$$
$$w(x,t) = z(y,u)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

Show that:

$$\partial_u z - \frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)z + z|z|^2\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2} = f(y,u)\,\frac{e^{-\frac{U}{2\gamma\chi}y+\frac{y^2}{2}}}{\gamma\chi^2}$$

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Ginzburg-Landau
Change of variables

$$w(x,t) = z(y,u)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

$$\partial_x w = \chi\partial_y w = \chi\left(\frac{\partial z}{\partial y} + z\left(\frac{U}{2\gamma\chi} - y\right)\right)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

$$\partial_{xx} w = \chi^2\left(\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\left(\frac{U}{2\gamma\chi} - y\right) - z + \left(\frac{\partial z}{\partial y} + z\left(\frac{U}{2\gamma\chi} - y\right)\right)\left(\frac{U}{2\gamma\chi} - y\right)\right)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

$$= \chi^2\left(\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\left(\frac{U}{2\gamma\chi} - y + \frac{U}{2\gamma\chi} - y\right) + z\left(\frac{U^2}{4\gamma^2\chi^2} + y^2 - \frac{U}{\gamma\chi}y - 1\right)\right)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

$$\partial_t w = \gamma\chi^2\partial_u z e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

Ginzburg-Landau
Change of variables

$$\partial_t w + U\partial_x w - (i\omega_0 + \mu_0 - \gamma\chi^4 x^2)w - \gamma\partial_{xx}w + w|w|^2 = f(x,t)$$

$$\Rightarrow \gamma\chi^2\partial_u z e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}} + U\chi\left(\frac{\partial z}{\partial y} + z\left(\frac{U}{2\gamma\chi} - y\right)\right)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}} - \left(i\omega_0 + \mu_0 - \gamma\chi^4\frac{y^2}{\chi^2}\right)z e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

$$-\gamma\chi^2\left(\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\left(\frac{U}{2\gamma\chi} - y + \frac{U}{2\gamma\chi} - y\right) + z\left(\frac{U^2}{4\gamma^2\chi^2} + y^2 - \frac{U}{\gamma\chi}y - 1\right)\right)e^{\frac{U}{2\gamma\chi}y-\frac{y^2}{2}}$$

$$+ z|z|^2 e^{\frac{3U}{2\gamma\chi}y-\frac{3y^2}{2}} = f(y,u)$$

$$\Rightarrow \partial_u z + \frac{U}{\gamma\chi}\left(\frac{\partial z}{\partial y} + z\left(\frac{U}{2\gamma\chi} - y\right)\right) - \left(\frac{i\omega_0 + \mu_0}{\gamma\chi^2} - y^2\right)z$$

$$-\left(\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\left(\frac{U}{\gamma\chi} - 2y\right) + z\left(\frac{U^2}{4\gamma^2\chi^2} + y^2 - \frac{U}{\gamma\chi}y - 1\right)\right) + z|z|^2\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2}$$

$$= f(y,u)\frac{e^{-\frac{U}{2\gamma\chi}y+\frac{y^2}{2}}}{\gamma\chi^2}$$

Ginzburg-Landau
Change of variables

$$\Rightarrow \partial_u z - \left(\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial y}\left(\frac{U}{\gamma\chi} - 2y - \frac{U}{\gamma\chi}\right) + z\left(\frac{U^2}{4\gamma^2\chi^2} + y^2 - \frac{U}{\gamma\chi}y - 1 + \frac{i\omega_0 + \mu_0}{\gamma\chi^2} - y^2 - \frac{U}{\gamma\chi}\left(\frac{U}{2\gamma\chi} - y\right)\right)\right) + \frac{1}{\gamma\chi^2} z|z|^2 e^{\frac{U}{\gamma\chi}y - y^2} = \frac{f(y,u)}{\gamma\chi^2}\ e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}$$

$$\Rightarrow \partial_u z - \left(\frac{\partial^2 z}{\partial y^2} - 2y\frac{\partial z}{\partial y} + z\left(-\frac{U^2}{4\gamma^2\chi^2} - 1 + \frac{i\omega_0 + \mu_0}{\gamma\chi^2}\right)\right) + \frac{1}{\gamma\chi^2} z|z|^2 e^{\frac{U}{\gamma\chi}y - y^2} = \frac{f(y,u)}{\gamma\chi^2}\ e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}$$

$$\Rightarrow \partial_u z - \frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right) z + z|z|^2 \frac{e^{\frac{U}{\gamma\chi}y - y^2}}{\gamma\chi^2} = f(y,u)\frac{e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}}{\gamma\chi^2}$$

Ginzburg-Landau Eigenvalues/eigenvectors

Show that the normal modes Eigenvalues: $z = \hat{z}(y)e^{\lambda u}$ of the linear opeartor:

$$\partial_u z - \frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)z = 0$$

Are:

$$\hat{z}_n = \zeta_n H_n(y), \qquad \zeta_n = (2^n n! \sqrt{\pi})^{-1/2}$$

$$\lambda_n = \frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 - 2n$$

The normalization coefficients ζ_n have been chosen such that:

$$\langle \hat{z}_n, \hat{z}_n \rangle = 1$$

$$\langle z_1, z_2 \rangle = \int_{-\infty}^{+\infty} \overline{z_1} z_2 e^{-y^2} dy$$

Hermite polynomials:

$$H_0(y) = 1, H_1(y) = 2y, \ldots$$

Ginzburg-Landau Eigenvalues/eigenvectors

$$\partial_u z - \frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{i\omega_0+\mu_0-\frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right) z = 0$$

Eigenvalues: $z = \hat{z}(y)e^{\lambda u}$

$$\lambda\hat{z} - \frac{\partial^2 \hat{z}}{\partial y^2} + 2y\frac{\partial \hat{z}}{\partial y} - \left(\frac{i\omega_0+\mu_0-\frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)\hat{z} = 0$$

$$-\frac{\partial^2 \hat{z}}{\partial y^2} + 2y\frac{\partial \hat{z}}{\partial y} - \underbrace{\left(\frac{i\omega_0+\mu_0-\frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 - \lambda\right)}_{=2n}\hat{z} = 0$$

$$\hat{z}_n = \zeta_n H_n(y), H_0 = \zeta_0, H_1 = 2\zeta_1 y, \ldots$$

$$\frac{i\omega_0+\mu_0-\frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 - \lambda_n = 2n \Rightarrow \lambda_n = \frac{i\omega_0+\mu_0-\frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 - 2n$$

Normalization coefficient such that: $\langle\hat{z}_n,\hat{z}_n\rangle = 1, \langle z_1, z_2\rangle = \int_{-\infty}^{+\infty}\overline{z_1}z_2 e^{-y^2}dy$

$$\langle\hat{z}_n,\hat{z}_n\rangle = 1 \Rightarrow \int_{-\infty}^{+\infty}\zeta_n H_n(y)\zeta_n H_n(y)e^{-y^2}dy = 1 \Rightarrow \zeta_n^2 = \left(\int_{-\infty}^{+\infty}H_n(y)H_n(y)e^{-y^2}dy\right)^{-1}$$

$$= (2^n n!\sqrt{\pi})^{-1}$$

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Ginzburg-Landau
Adjoint operator

Let:

$$\mathcal{M}z = -\frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)z$$

Considering the scalar-product:

$$\langle z_1, z_2 \rangle = \int_{-\infty}^{+\infty} \overline{z_1} z_2 e^{-y^2} dy$$

Show that the adjoint operator $\widetilde{\mathcal{M}}$ verifying

$$\langle z_1, \mathcal{M}z_2 \rangle = \left\langle \widetilde{\mathcal{M}}z_1, z_2 \right\rangle$$

Is:

$$\widetilde{\mathcal{M}}z = -\frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)z$$

Show that $\mathcal{M}$ is normal ($\widetilde{\mathcal{M}}\mathcal{M} = \mathcal{M}\widetilde{\mathcal{M}}$), so that the normal mode basis is orthogonal with respect to $\langle z_1, z_2 \rangle$.

Ginzburg-Landau
Adjoint operator

$$\langle z_1, \mathcal{M} z_2 \rangle = \int_{-\infty}^{+\infty} \left(\bar{z_1} 2y \frac{\partial z_2}{\partial y} e^{-y^2} - \bar{z_1} \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \right) z_2 e^{-y^2} - \bar{z_1} \frac{\partial^2 z_2}{\partial y^2} e^{-y^2} \right) dy$$

$$= \int_{-\infty}^{+\infty} \left(-\overline{\frac{\partial}{\partial_y} \left(z_1 2y e^{-y^2} \right)} z_2 - \overline{\left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \right) z_1} z_2 e^{-y^2} \right.$$

$$\left. - \overline{\frac{\partial^2}{\partial y^2} \left(z_1 e^{-y^2} \right)} z_2 \right) dy$$

Ginzburg-Landau
Adjoint operator

$$\langle z_1, \mathcal{M} z_2 \rangle =$$

$$= \int_{-\infty}^{+\infty} \left(-\overline{\left(\frac{\partial z_1}{\partial y} 2ye^{-y^2} + z_1 2e^{-y^2} - 2yz_1 2ye^{-y^2} \right)} z_2 \right.$$

$$- \overline{\left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \right)} z_1 z_2 e^{-y^2}$$

$$\left. - \overline{\left(\frac{\partial^2 z_1}{\partial y^2} e^{-y^2} - 2y \frac{\partial z_1}{\partial y} e^{-y^2} - 2z_1 e^{-y^2} - 2y \frac{\partial z_1}{\partial y} e^{-y^2} + 4y^2 z_1 e^{-y^2} \right)} z_2 \right) dy$$

Ginzburg-Landau
Adjoint operator

$$\langle z_1, \mathcal{M} z_2 \rangle = \int_{-\infty}^{+\infty} \left(\overline{\left(\frac{\partial z_1}{\partial y} 2y \right)} z_2 - \overline{\left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \right)} z_1 z_2 - \overline{\frac{\partial^2 z_1}{\partial y^2}} z_2 \right) e^{-y^2} dy$$
$$= \langle \widetilde{\mathcal{M}} z_1, z_2 \rangle$$

$$\widetilde{\mathcal{M}} z_1 = 2y \frac{\partial z_1}{\partial y} - \left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \right) z_1 - \frac{\partial^2 z_1}{\partial y^2}$$

$$\mathcal{M} z = 2y \frac{\partial z}{\partial y} - \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \right) z - \frac{\partial^2 z}{\partial y^2}$$

Normal operator: $\widetilde{\mathcal{M}}\mathcal{M} = \mathcal{M}\widetilde{\mathcal{M}}$

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Ginzburg-Landau
Adjoint eigenvectors

Show that the normal modes: $z = \tilde{z}(y)e^{\lambda u}$ of the linear opeartor:

$$\partial_u z - \frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{-i\omega_0+\mu_0-\frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)z = 0$$

Are:

$$\tilde{z}_n = \xi_n H_n(y), \qquad \xi_n = (2^n n!\sqrt{\pi})^{-1/2}$$

$$\lambda_n = \frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 - 2n$$

The normalization coefficients ξ_n have been chosen such that:

$$\langle \tilde{z}_n, \hat{z}_n \rangle = 1$$

Ginzburg-Landau
Adjoint eigenvectors

$$\lambda\tilde{z} + \widetilde{\mathcal{M}}\tilde{z} = 0$$

$$\lambda\tilde{z} + 2y\frac{\partial\tilde{z}}{\partial y} - \left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2}\right)\tilde{z} - \frac{\partial^2\tilde{z}}{\partial y^2} = 0$$

$$2y\frac{\partial\tilde{z}}{\partial y} - \left(\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - \lambda\right)\tilde{z} - \frac{\partial^2\tilde{z}}{\partial y^2} = 0$$

$$\frac{-i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - \lambda_n = 2n, \tilde{z}_n = \xi_n H_n(y)$$

Normalization coeffcient:

$$\langle\tilde{z}_n, \hat{z}_n\rangle = 1 \Rightarrow \int_{-\infty}^{+\infty} \xi_n H_n(y)\zeta_n H_n(y) e^{-y^2} dy = 1 \Rightarrow \xi_n = \frac{1}{\zeta_n \int_{-\infty}^{+\infty} H_n(y) H_n(y) e^{-y^2} dy} = \zeta_n$$

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Ginzburg-Landau WNL

There exists $\mu_0 = \mu_0^c$ so that the least-damped normal is marginal:

$$\lambda_0 = \frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 \Rightarrow \frac{\mu_0^c - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1 = 0$$

We now consider a slightly supercritical parameter:

$$\mu_0 = \mu_0^c + \delta' = \mu_0^c + \epsilon\delta$$

With $\epsilon \ll 1$and $\delta = O(1)$.
Considering the following forcing:

$$f(x,t) = E'\delta\left(y - y_f\right)e^{i\omega_f t}$$

$$\omega_f = \omega_0 + \Omega', \Omega' = \epsilon\Omega$$

$$E' = \epsilon^{\frac{3}{2}}E$$

If

$$z = \epsilon^{\frac{1}{2}}z'$$

What is the equation verified by z'?

Ginzburg-Landau
WNL

$$\partial_u z - \frac{\partial^2 z}{\partial y^2} + 2y\frac{\partial z}{\partial y} - \left(\frac{i\omega_0 + \mu_0 - \frac{U^2}{4\gamma}}{\gamma\chi^2} - 1\right)z + z|z|^2\frac{e^{\frac{U}{\gamma\chi}y - y^2}}{\gamma\chi^2}$$

$$= E'\delta(y - y_f)e^{\frac{i\omega_f u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}}{\gamma\chi^2}$$

$$u = \gamma\chi^2 t$$

$$\Rightarrow \partial_u z' - \frac{\partial^2 z'}{\partial y^2} + 2y\frac{\partial z'}{\partial y} - \left(\frac{i\omega_0 + \epsilon\delta}{\gamma\chi^2}\right)z' + \epsilon z'|z'|^2\frac{e^{\frac{U}{\gamma\chi}y - y^2}}{\gamma\chi^2}$$

$$= \epsilon E\delta(y - y_f)e^{\frac{i\omega_0 u}{\gamma\chi^2}}e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}}{\gamma\chi^2}$$

Ginzburg-Landau WNL

We look for a solution under the form:

$$z' = z_0'(u,\tau) + \epsilon z_1'(u,\tau) + \cdots$$
$$\tau = \epsilon u$$

What are the equations governing $z_0'(u,\tau)$ and $z_1'(u,\tau)$?
Show that

$$z_0'(u,\tau) = A(\tau) e^{i\frac{\omega_0}{\gamma\chi^2}u} \hat{z}_0(y)$$

is an acceptable solution
and that $z_1'(u,\tau)$ remains bounded if:

$$\frac{dA}{d\tau} = \frac{\delta}{\gamma\chi^2} A - \left\langle \tilde{z}_0, \hat{z}_0 |\hat{z}_0|^2 \frac{e^{\frac{U}{\gamma\chi}y - y^2}}{\gamma\chi^2} \right\rangle A|A|^2 + E e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}} \tilde{z}_0(y_f) \frac{e^{-\frac{U}{2\gamma\chi}y_f - \frac{y_f^2}{2}}}{\gamma\chi^2}$$

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Ginzburg-Landau
WNL

$$z' = z_0'(u,\tau) + \epsilon z_1'(u,\tau) + \cdots \Rightarrow \partial_u z' = \frac{\partial z_0'}{\partial u} + \epsilon\left(\frac{\partial z_0'}{\partial \tau} + \frac{\partial z_1'}{\partial u}\right) + \cdots$$

$$\Rightarrow \frac{\partial z_0'}{\partial u} + \epsilon\left(\frac{\partial z_0'}{\partial \tau} + \frac{\partial z_1'}{\partial u}\right) + 2y\left(\frac{\partial z_0'}{\partial y} + \epsilon\frac{\partial z_1'}{\partial y}\right) - \left(\frac{i\omega_0 + \epsilon\delta}{\gamma\chi^2}\right)(z_0' + \epsilon z_1') - \left(\frac{\partial^2 z_0'}{\partial y^2} + \epsilon\frac{\partial^2 z_1'}{\partial y^2}\right)$$

$$+ \epsilon z_0'|z_0'|^2 \frac{e^{\frac{U}{\gamma\chi}y - y^2}}{\gamma\chi^2} = \epsilon E\delta\left(y - y_f\right)e^{\frac{i\omega_0 u}{\gamma\chi^2}}e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}}{\gamma\chi^2}$$

Order 1:

$$\frac{\partial z_0'}{\partial u} - \frac{\partial^2 z_0'}{\partial y^2} + 2y\frac{\partial z_0'}{\partial y} - \frac{i\omega_0}{\gamma\chi^2}z_0' = 0$$

Order ϵ:

$$\left(\frac{\partial z_0'}{\partial \tau} + \frac{\partial z_1'}{\partial u}\right) - \frac{\partial^2 z_1'}{\partial y^2} + 2y\frac{\partial z_1'}{\partial y} - \frac{i\omega_0}{\gamma\chi^2}z_1' - \frac{\delta}{\gamma\chi^2}z_0' + z_0'|z_0'|^2\frac{e^{\frac{U}{\gamma\chi}y - y^2}}{\gamma\chi^2}$$

$$= E\delta\left(y - y_f\right)e^{\frac{i\omega_0 u}{\gamma\chi^2}}e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y + \frac{y^2}{2}}}{\gamma\chi^2}$$

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Order 1:

$$\frac{\partial z_0'}{\partial u} + 2y\frac{\partial z_0'}{\partial y} - \frac{i\omega_0}{\gamma\chi^2}z_0' - \frac{\partial^2 z_0'}{\partial y^2} = 0$$

$$\Rightarrow z_0' = A(\tau)e^{i\frac{\omega_0}{\gamma\chi^2}u}\hat{z}_0(y)$$

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Order ϵ:

$$\left(\frac{\partial z_0'}{\partial \tau}+\frac{\partial z_1'}{\partial u}\right)-\frac{\partial^2 z_1'}{\partial y^2}+2y\frac{\partial z_1'}{\partial y}-\frac{i\omega_0}{\gamma\chi^2}z_1'-\left(\frac{\delta}{\gamma\chi^2}\right)z_0'+z_0'|z_0'|^2\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2}$$

$$=E\delta\left(y-y_f\right)e^{\frac{i\omega_0 u}{\gamma\chi^2}}e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y+\frac{y^2}{2}}}{\gamma\chi^2}$$

$$\Rightarrow \frac{\partial z_1'}{\partial u}-\frac{\partial^2 z_1'}{\partial y^2}+2y\frac{\partial z_1'}{\partial y}-\frac{i\omega_0}{\gamma\chi^2}z_1'$$

$$=-\frac{dA}{d\tau}e^{i\frac{\omega_0}{\gamma\chi^2}u}\hat{z}_0+\frac{\delta}{\gamma\chi^2}A\hat{z}_0e^{i\frac{\omega_0}{\gamma\chi^2}u}-A|A|^2\hat{z}_0|\hat{z}_0|^2e^{i\frac{\omega_0}{\gamma\chi^2}u}\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2}$$

$$+E\delta\left(y-y_f\right)e^{\frac{i\omega_0 u}{\gamma\chi^2}}e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y+\frac{y^2}{2}}}{\gamma\chi^2}$$

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WNL

$$\Rightarrow \left\langle \tilde{z}_0, -\frac{dA}{d\tau}\hat{z}_0 + \frac{\delta}{\gamma\chi^2}A\hat{z}_0 - A|A|^2\hat{z}_0|\hat{z}_0|^2\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2} + E\delta(y-y_f)e^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\frac{e^{-\frac{U}{2\gamma\chi}y+\frac{y^2}{2}}}{\gamma\chi^2}\right\rangle = 0$$

$$\Rightarrow -\frac{dA}{d\tau}\langle\tilde{z}_0,\hat{z}_0\rangle + \frac{\delta}{\gamma\chi^2}A\langle\tilde{z}_0,\hat{z}_0\rangle - A|A|^2\left\langle\tilde{z}_0,\hat{z}_0|\hat{z}_0|^2\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2}\right\rangle$$
$$+ Ee^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\left\langle\tilde{z}_0,\delta(y-y_f)\frac{e^{-\frac{U}{2\gamma\chi}y+\frac{y^2}{2}}}{\gamma\chi^2}\right\rangle = 0$$

$$\frac{dA}{d\tau} = \frac{\delta}{\gamma\chi^2}A - \left\langle\tilde{z}_0,\hat{z}_0|\hat{z}_0|^2\frac{e^{\frac{U}{\gamma\chi}y-y^2}}{\gamma\chi^2}\right\rangle A|A|^2 + Ee^{\frac{i\epsilon\Omega u}{\gamma\chi^2}}\tilde{z}_0(y_f)\frac{e^{-\frac{U}{2\gamma\chi}y_f-\frac{y_f^2}{2}}}{\gamma\chi^2}$$

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Ginzburg-Landau
WNL

Show that the physical solution is:

$$w(x,t) = \zeta_0 C'(t) e^{i\omega_f t} e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\frac{dC'}{dt} = (-i\Omega' + \delta')C' - \left\langle \tilde{z}_0, \hat{z}_0 |\hat{z}_0|^2 e^{\frac{U}{\gamma\chi}y - y^2} \right\rangle C'|C'|^2 + \xi_0 e^{-\frac{U}{2\gamma}x_f - \frac{\chi^2 x_f^2}{2}} E'$$

And that the saturation amplitude of the limit-cycle oscillations is:

$$\zeta_0 \max_t |C'(t)| = 2^{1/4} e^{-\frac{U^2}{16\gamma^2\chi^2}} \sqrt{\delta'}$$

We remind that in the case of the naïve approach:

$$w(x,t) = \zeta C'(t) e^{i\omega_f t} e^{\frac{U}{2\gamma}x - \frac{\chi^2 x^2}{2}}$$

$$\frac{dC'}{dt} = (-i\Omega' + \delta')C' - < \widetilde{w}_c, \widehat{w}_c |\widehat{w}_c|^2 > C'|C'|^2 + \xi e^{-\frac{U}{2\gamma}x_f - \frac{\chi^2 x_f^2}{2}} E'$$

Show that the saturation amplitude $\zeta \max_t |C'(t)|$ exhibits the same value.

Differences between the two approaches are expected at the next order in fact.

Ginzburg-Landau
WNL

$$\left\langle \tilde{z}_0, \hat{z}_0 |\hat{z}_0|^2 e^{\frac{U}{\gamma\chi}y - y^2} \right\rangle = \xi_0 \zeta_0^3 \int_{-\infty}^{+\infty} e^{\frac{U}{\gamma\chi}y - y^2} e^{-y^2} dy = \pi^{-1}\chi \int_{-\infty}^{+\infty} e^{\frac{U}{\gamma}x - 2\chi^2 x^2} dx$$

$$\zeta_0 \max_t |C'(t)| = \frac{\zeta_0}{\sqrt{\left\langle \tilde{z}_0, \hat{z}_0 |\hat{z}_0|^2 e^{\frac{U}{\gamma\chi}y - y^2} \right\rangle}} \sqrt{\delta'} = \frac{\pi^{-\frac{1}{4}}}{\pi^{-\frac{1}{2}}\chi^{\frac{1}{2}}\sqrt{\int_{-\infty}^{+\infty} e^{\frac{U}{\gamma}x - 2\chi^2 x^2} dx}} \sqrt{\delta'}$$

$$= \frac{\pi^{\frac{1}{4}}}{\chi^{\frac{1}{2}} \left(\frac{e^{\frac{U^2}{8\gamma^2\chi^2}} \pi^{\frac{1}{2}}}{\sqrt{2}\chi} \right)^{\frac{1}{2}}} \sqrt{\delta'} = 2^{1/4} e^{-\frac{U^2}{16\gamma^2\chi^2}} \sqrt{\delta'}$$

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The Ginzburg-Landau eq.

WNL

$$< \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 > = \int_{-\infty}^{+\infty} \xi e^{-\frac{U}{2\gamma}x-\frac{\chi^2 x^2}{2}} \zeta^3 e^{\frac{3U}{2\gamma}x-\frac{3\chi^2 x^2}{2}} dx = \xi\zeta^3 \int_{-\infty}^{+\infty} e^{\frac{U}{\gamma}x-2\chi^2 x^2} dx$$

$$= \frac{\chi}{\zeta \pi^{\frac{1}{2}}} \zeta^3 \int_{-\infty}^{+\infty} e^{\frac{U}{\gamma}x-2\chi^2 x^2} dx$$

$$\zeta \max_t |C'(t)| = \frac{\zeta}{\sqrt{< \widetilde{w}_c, \widehat{w}_c|\widehat{w}_c|^2 >}} \sqrt{\delta'} = \frac{\pi^{\frac{1}{4}}}{\chi^{\frac{1}{2}} \sqrt{\int_{-\infty}^{+\infty} e^{\frac{U}{\gamma}x-2\chi^2 x^2} dx}} \sqrt{\delta'} = \frac{\pi^{\frac{1}{4}}}{\chi^{\frac{1}{2}} \left(\frac{e^{\frac{U^2}{8\gamma^2\chi^2}} \pi^{\frac{1}{2}}}{\sqrt{2}\chi} \right)^{\frac{1}{2}}} \sqrt{\delta'}$$

$$= 2^{1/4} e^{-\frac{U^2}{16\gamma^2\chi^2}} \sqrt{\delta'} = \frac{2^{\frac{1}{4}}}{e^{\frac{U^2}{16\gamma^2\chi^2}}} \sqrt{\delta'}$$